Second-Order Representation of Signals

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1 Introduction

During the past decades much effort has been devoted to the adequate representation of signals in various fields such as acoustics and (bio-)electrical engineering. Among others, this resulted in a number of second order signal representations, each one with its own particular characteristics and special domain of application. A number of these functionals can be met regularly in the literature dealing with the problems of localization and identification of signal sources, both in technical and biological applications. Especially those second order functionals which have time and/or frequency, in the various possible configurations, as arguments have shown to be of particular interest: e.g. the lagged product function, the ambiguity function (Woodward 1953, Rihaczek 1969), the bispectrum, the Wigner-distribution (Wigner 1932, Claasen and Mecklenbräuker 1980), and the Rihaczek-CoSTID (Johannesma and Aertsen 1983). In the present paper we present a general scheme providing the formal relations between these functionals.

2 Theory

Throughout this paper we will use the analytic signal $\xi(t)$ instead of the real signal $x(t)$, since the former leads to more elegant formulations of important signal characteristics. Based on the analytic signal the lagged product function Π is defined as

$$\Pi(\tau, t; a) = \xi^*[t + (a - \frac{1}{2})\tau]\xi[t + (a + \frac{1}{2})\tau] \qquad -\frac{1}{2} \leqslant a \leqslant \frac{1}{2} \tag{1}$$

with * denoting complex conjugation.

From the product function the ambiguity function $\Delta(\tau, \nu)$ is derived by Fourier transformation with respect to t. The time-frequency density $\Xi(\omega, t)$ is the Fourier transform with respect to τ while the bispectrum $\Gamma(\omega, \nu)$ is produced by Fourier transformation with respect to both τ and t. This leads to the following relations:

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- ambiguity function

$$\Delta(\tau, \nu; a) = \int dt\, e^{-i\nu t}\, \Pi(\tau, t; a) \quad (2)$$

- time frequency density or CoSTID

$$\Xi(\omega, t; a) = \int d\tau\, e^{-i\omega t}\, \Pi(\tau, t; a) \quad (3)$$

- bispectrum

$$\Gamma(\omega, \nu; a) = \int d\tau\, e^{-i\omega\tau} \int dt\, e^{-i\nu t}\, \Pi(\tau, t; a) \quad (4)$$

$$= \xi^* [\omega - (a + \tfrac{1}{2})\nu]\, \xi[\omega - (a - \tfrac{1}{2})\nu] \quad (5)$$

where $\xi(\omega)$ denotes the Fourier transform of $\xi(t)$, and all integrals have to be taken from $-\infty$ to $+\infty$. A rigorous and detailed discussion on the relations between these functionals and their respective properties, as well as on the influence of the parameter a is outside the scope of the present paper, and will be considered elsewhere (Aertsen et al. in prep.). Figure 1 summarizes the relations between the functionals introduced above. By appropriate choice of the parameter a most second order functionals proposed in the literature can be reached. By taking $a = -\frac{1}{2}$ one obtains a quadruplet of functionals with the (complex) Rihaczek-function $\Xi(\omega, t; -\frac{1}{2})$ (Rihaczek 1968; see also companion paper) as a cornerstone, the *Rihaczek quadruplet*; $a = 0$ leads to the *Wigner quadruplet* with the (real) Wigner distribution $\Xi(\omega, t; 0)$ at the corresponding position.

3 Representation

Visualization of a complex second-order functional using standard graphical techniques (3-D, gray-coding) leads to a doublet of pictures: either real and imaginary part, or amplitude and phase. The two can be integrated by using a color mapping of the complex plane (Johannesma et al. 1981).

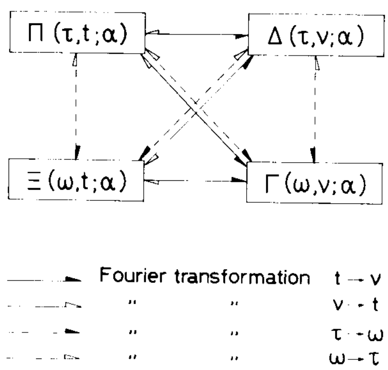

Fig. 1. Generalized diagram of relations between different second order signal representations

Gray-coding representations of the functionals from the two quadruplets have been presented at the conference for various signals (different tone combinations, chirp). From these and other examples the following conclusions can be drawn:

- a shift in frequency and/or time leads to a translation in $\equiv$, a complex modulation in Δ and, in general, a mixture of these in Π and Γ;
- the influence of a is virtually absent in Δ, trivially present in Π and Γ, whereas in $\equiv$ the influence is pronounced and more intricate;
- comparison of the Wigner distribution $\equiv(\omega, t; 0)$ and the Rihaczek-CoSTID $\equiv(\omega, t; \pm\frac{1}{2})$ shows the following (see also Claasen and Mecklenbräuker (1980), Johannesma et al. (1981), and companion paper): (see also Fig. 2):

$\equiv(\omega, t; 0)$	$\equiv(\omega, t; \pm\frac{1}{2})$
– real	– complex
– non-factorable	– factorable
– minimum spread	– larger spread
– cross terms on "diagonal"	– cross terms "off-diagonal"

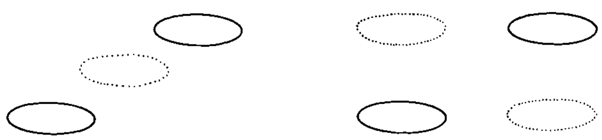

Fig. 2. Comparison of the Wigner distribution $\equiv(\omega, t; 0)$ (left side) and the Rihaczek-CoSTID $\equiv(\omega, t; \pm\frac{1}{2})$

Which functional is optimal in general will depend on the special question of interest. On the whole the CoSTID $\equiv(\omega, t; a)$ appears to be the first natural candidate because of its intuitive base and close relation to auditory perception [e.g. Altes (1980), Aertsen and Johannesma (1981), Hermes et al. (1981)].

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